A Heavy Glueball with Color-Singletness Restriction at Finite Temperature

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Abstract

We show that a heavy glueball (much heavier than that studied by others which is in the range of 1-2 GeV) is generated in a pure gluon plasma when color-singletness condition is imposed on the partition function at finite temperature. This confirms Abbas's recent prediction (hep-ph/9504430) of the existence of a heavy glueball within the framework of the early universe scenario.

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The study of the strong interaction physics within the farmework of Quantum Chromodynamics(QCD) [1] clearly indicates the existence of bound states of gluons — the so called glueballs. Much work has been done on glueballs, but all the theoretical calculations [2,3] estimate the glueballs within the mass range of a few GeV. Recently Abbas [4] has shown that the idea of two distinct phase transition temperatures [5] in QGP can produce very massive glueballs ($\geq 45GeV$) — which he suggested as the new dark matter (DM) candidate. Here we wish to look at his prediction of the existence of a heavy glueball in another framework.

Some work has been done [6-11] by imposing color-singletness restriction on the total partition function of the quark - gluon system to obtain the thermodynamic quantities and the consequence of the $SU(3)$ color- singletness restriction on the quark - gluon deconfinement phase diagram was discussed [8] arround the critical temperature T_c. It was pointed out that the color-singletness constraint gives a finite volume correction automatically. Auberson et al. [6] and Savatier [7] have suggested a formalism to project out the allowed color-singlet states of the $SU(N)$ color group in the partition function of the quark - gluon system.

Our aim in this letter is to discuss the importance of the $SU(3)$ color- singletness restriction on the pure gluonic system (glueball). We will show that if we do not constrain the gluon partition function to be color-singlet then there exists only one solution for a glueball bag radius upto a certain temperature (say T_s). Above T_s there is no valid solution. But imposing the color-singletness restriction on the partition function of the gluonic system, one finds that for $T < T_s$ we have only one valid solution for the glueball. For $T > T_c$ there are no valid solutions. Interestingly for $T_s < T < T_c$ we obtain two solutions for the radius of the glueballs. One of the solution gives a very massive metastable state. This arises due to the finite volume correction of the color-singletness

restriction. Hence we are cofirming Abbas's recent prediction [4] for the existence of a heavy glueball arising in cosmological considerations.

The grand canonical partition function [6-11] for a quark-gluon system is given by

$$\mathcal{Z}(\beta, V) = Tr \left(\hat{\mathcal{P}} \; e^{-\beta \hat{H}} \right) \;, \tag{1}$$

where $\beta = \frac{1}{T}$ is the inverse of the temperature, V is the volume and $\hat{H}$ is the total Hamiltonian of the physical system. $\hat{\mathcal{P}}$ is the projection operator to give a particular configuration (e.g. Angular momentum, Number, Parity, Color etc.) allowed by the system.

The projection operator for a particular j^{th} irreducible representation of a symmetry group $\mathcal{G}$ can be written as [12]

$$\hat{\mathcal{P}}_j \;=\; d_j \int_{\mathcal{G}} d\mu(g) \; \chi_j^{\star}(g) \; \hat{U}(g) \;, \tag{2}$$

where $\hat{U}(g)$is the unitary representation of the group $\mathcal{G}$ in the Hilbert Space $\mathcal{H}$, d_j and χ_j , are the dimension and the character of the irreducible representation respectively, $d\mu(g)$ is the normalized Haar measure in $\mathcal{G}$.

We will derive the color-singlet partition function for the gluonic system with $SU(3)$ color symmetry group. So for $SU(3)$ color - singlet configuration $d_j \;=\; 1$ and hence $\chi_j \;=\; 1$. Since the Hilbert Space is spanned by the gluons only, so $\mathcal{H} \;=\; \mathcal{H}_g$ and the partition function for the gluonic system becomes (for details see ref.[6-8]),

$$\mathcal{Z}(\beta, V) \;=\; \int_{SU(3)} d\mu(g) \; Tr \left(\; \hat{U}_g(g) \; e^{-\beta \hat{H}_g} \right) \;. \tag{3}$$

Where $\hat{H}_g$ is the total hamiltonian for gluons.

The normalised Haar measure for $SU(3)$ color group is given by [6,7]

$$\int d\mu(g) \; = \; \frac{1}{24\pi^2} \int_{-\pi}^{\pi} d\theta_1 \; d\theta_2 \prod_{i<j}^{3} \Big(2 \; Sin\frac{1}{2}(\theta_i - \theta_j)\Big)^2 \tag{4}$$

Performing the trace operation [6-8] in equation(3) the partition function becomes,

$$\mathcal{Z}(\beta, V) \; = \; \int_{SU(3)} d\mu(g) \; e^{\Theta} \; , \tag{5}$$

where

$$\Theta \quad = \quad \sum_{\alpha} \sum_{k=1}^{\infty} \Big[\; \frac{1}{k} \; \chi_{adj}(g^k) \; e^{-k\beta\epsilon_g^{\alpha}} \; \Big] \; , \tag{6}$$

and $\chi_{adj}(g^k) \; = \; 2 + \sum_{i<j}^{3} \; \cos \; k(\theta_i \; - \; \theta_j)$ is the character of the adjoint representation of the symmetry $SU(3)$ color group related to the class paarmeter θ_i such that $\sum_{i}^{3} \; \theta_i \; = \; 0 \;\; (mod \; 2\pi)$.

The summation over the index k arises due to the trace operation whereas over the index α is due to the discrete single particle states of gluons with energy ϵ_g^{α}.
In a reasonably dense system we can replace $\sum_{\alpha}$ by $2 \; \int \; \rho(\epsilon_g) \; d\epsilon_g$ in Eq.(6) in the large volume limit. Here $\rho(\epsilon_g) \; = \; \frac{V{\epsilon_g}^2}{2\pi^2}$ is the gluon single-particle density of states with V being the volume and the factor 2 in the integration is understood for gluons.
After doing the integration in Eq.(6), we get

$$\Theta \; = \; \frac{2\pi^2 V}{45\beta^3} \; + \; \frac{4V}{\pi^2\beta^3} \sum_{i<j}^{3} u(\theta_i \; - \; \theta_j) \; , \tag{7}$$

where,

$$\begin{aligned} u(\theta_i \; - \; \theta_j) \; &= \; \sum_{k=1}^{\infty} \frac{\cos \; k(\theta_i \; - \; \theta_j)}{k^4} \\ &= \; \frac{\pi^4}{90} - \frac{\pi^2}{12}(\theta_i \; - \; \theta_j)^2 \Big\{1 - \frac{|\theta_i \; - \; \theta_j|}{2\pi}\Big\}^2 \end{aligned} \tag{8}$$

such that $(|\theta_i \; - \; \theta_j| \; < \; 2\pi)$. Now, using this expression for Θ and integrating the group integration in the saddle point approximation [6-8], from Eq.(5) finally we get the

color-singlet partition function for the gluons as

$$\mathcal{Z}(\beta, V) \;=\; \frac{\sqrt{3}}{3\pi}\left[\,\frac{2V}{\beta^3}\,\right]^{-4}\;\exp\;\left[\,\frac{8\pi^2 V}{45\beta^3}\,\right] \tag{9}$$

This color-singlet partition function for gluons is used to describe the thermodynamic properties of the glueball in a bag like picture.

Now to understand the effect of the finite volume correction generated by color-singletness constraint on the partition function, we consider the gluons as confined in a bag of radius R with a pressure constant B. On including the zero temperature bag term energy [13] $(BV \;+\; C/R)$ we write respectively the free energy and the energy of the color-singlet glueball in a bag as,

$$\begin{aligned} F(T,V) \;&=\; -\,T\;\ln \mathcal{Z}(T,V)\;+\;BV\;+\;\frac{C}{R} \\ &=\; BV\;+\;\frac{C}{R}\;-\;KVT^4 \\ &+\; 4T\ln\left[2VT^3\right]\;+\;T\ln\left[\sqrt{3}\pi\right] \end{aligned} \tag{10}$$

and

$$\begin{aligned} E(T,V) \;&=\; T^2\,\frac{\partial}{\partial T}\Big(\ln \mathcal{Z}(T,V)\Big)\;+\;BV\;+\;\frac{C}{R} \\ &=\; BV\;+\;\frac{C}{R}\;+\;3KVT^4\;-\;12T \end{aligned} \tag{11}$$

where

$$K \;=\; \frac{8\pi^2}{45} \tag{12}$$

and C is a positive constant which depends on the internal quantum numbers of the bag.

For the stability of the bag[13], the pressure balance condition $P \;=\; -\left(\frac{\partial F}{\partial V}\right)_T \;=\; 0$ gives

$$P \;=\; \frac{C}{4\pi R^4}\;-\;B\;+\;KT^4\;-\;\frac{3T}{\pi R^3}\;=\;0\;. \tag{13}$$

It is to be noted that the last term in l.h.s. of Eq.(13) is the finite volume correction due to color-singletness restriction.

Using Eq.(13) and eleminating the T dependence of Eq.(11), we get the average bag internal energy corresponding to the equilibrium state as,

$$E \;=\; 4BV \tag{14}$$

Which satisfies the virial theorem.

Now, with $x \;=\; \frac{1}{R}$, the equation (13) becomes,

$$x^4 \;=\; ax^3 \;+\; g \quad , \tag{15}$$

$$\textit{where} \quad a \;=\; \frac{12T}{C}, \quad g \;=\; \frac{4\pi K}{C}\left(T_s^4 \;-\; T^4\right) \;\; \textit{and} \;\; T_s \;=\; \left(\frac{B}{K}\right)^{\frac{1}{4}} \tag{16}$$

We will see that this T_s behaves as a transition temperature at which two bag solutions appears indicating a metastable state for a glueball with heavy mass. These two solutions arises due to the $SU(3)$ color-singletness restriction.

Now, we solve Eq.(15) graphically. From the intersection of the two curves $y = x^4$ and $y = ax^3 + g$ (as in Fig.1.) we get two solutions (solid circles) of Eq.(15) for a particular temperature between $T_s \leq T < T_c$ and only one solution for $T < T_s$ and at $T = T_c$. In Fig.1. the graphical solution of Eq.(15) is shown for four different temperatures $T = 210$ MeV ($< T_s$), $T = T_s = 217.218$ MeV, $T = 240$ MeV and $T = T_c = 253.3$ MeV when $B^{\frac{1}{4}} = 250$ MeV. The unphysical bag solutions corresponding to negative x are excluded.

(i) For $T < T_s$, there is only one solution of the bag (curve (b) as in Fig.1.) corresponds to a stable glueball with very low mass.

(ii) When $T = T_s$, $g = 0$.

Two curves (curve (a)) $y = x^4$ and (curve (c)) $y = ax^3 + g$ intersects at $x = 0$ giving a solution at $R = \infty$ and at $x = \frac{12T_s}{C}$ leading to a finite bag radius $R = \frac{C}{12T_s}$. Note that here the large mass glueball just starts forming and appears at infinity.

(iii) When $T > T_s$, the two solutions approaches each other (see the intersection of curve (a) and curve (d) of Fig.1.) with one solution for smaller R and the other with larger R . This indicates that at T_s, transition occurs from a small stable bag to a larger metastable bag of glueball. The numerical values of the solutions shall be discussed shortly.

(iv) As the two solutions approaches each other with increasing temperature, so there must be a critical temperature (T_c) at which both the solutions meet at one critical point (see curve (a) and curve (e) in Fig.1.), where two curves (a) and (e) touch tangentially with the same slope as $x(T_c) = \frac{9T_c}{C}$. Hence that the critical radius of the bag of the gluons is $R(T_c) = \frac{C}{9T_c}$ above which no bag solutions exists. This critical temperature can be obtained from Eq.(15) as

$$T_c = \frac{T_s}{\left[1 - \frac{2187}{4\pi K C^3} \right]^{1/4}} \tag{17}$$

provided $C > (2187/4\pi K)^{1/3}$.

Here it is to be noted that at equilibrium, one can calculate energy of the system either from Eq.(11) or Eq.(14) corresponding to different temperature T.

If we ignore the color-singletness restriction on the partition function, one obtains the total energy and the free energy of the gluon bag as respectively,

$$E = BV + \frac{C}{R} + 3KVT^4 \tag{18}$$

$$F \; = \; BV \; + \; \frac{C}{R} \; - \; KVT^4 \tag{19}$$

So the pressure balance condition gives the expression for R as

$$R \; = \; \left(\frac{C}{4\pi K} \right)^{\frac{1}{4}} \left(T_s^4 \; - \; T^4 \right)^{-\frac{1}{4}} \tag{20}$$

From this equation we see that for $T < T_s$, there is only one finite solution for a bag. For $T \; = \; T_s$, the bag solution is at $R = \infty$ and for $T > T_s$ there is no physically real solution of the bag. So the appearence of large metastable glueball is because of the $SU(3)$ color-singletness restriction in the pure gluonic plasma.

Now we give numerical values for a particular case (see Table 1.). For $B^{\frac{1}{4}} = 250$ MeV , we get the temperature $T_s = 217.218$ MeV from Eq.(16) and the critical temperature $T_c = 253.3$ MeV from Eq.(17). For a temperature say $T = 210$ MeV ($< T_s$) , we get only one real solution ($R = 0.45$ fm corresponding to the intersection of the curves (a) and (b) in Fig.1.) of the gluon bag with mass 0.79 GeV by solving Eq.(15). As T exceeds T_s, we get two real solutions for the bag, one with a lower radius and the other with higher radius as shown in Fig.2. For different temperatures between $T_s < T < T_c$ ($218 - 250$ MeV) , the low mass state and the high mass state with different R values are shown in Table 1. We notice that the lower radius solution is gradually increasing when the larger radius solution decreases fast with increasing temperature. At last at $T = T_c = 253.3$ MeV, we get only one solution with $R \sim 0.5$ fm and $M \sim 1$ GeV .

Similarly we saw that for $B^{\frac{1}{4}} = 400$ MeV , at $T = 348$ MeV ($> T_s = 347.55$ MeV) , the massive state (952 GeV) having larger radius ($R = 2.57$ fm) and the low mass state ($M = 1.27$ GeV) has a radius $R = 0.28$ fm . At $T = T_c = 405.282$ MeV, we get only one solution with $R = 0.32$ fm and $M = 1.9$ GeV and at $T = 355$ MeV, the higher mass state of 45 GeV has radius 0.93 fm .

Hence by studying the color projected and the unprojected partition functions of the gluonic system as well as their thermodynamic quantities we see that, the bag radius of the system diverges with temperature upto $T = T_s$ for the color unprojected case. Whereas the finite volume correction to the bag is automatically generated due to the $SU(3)$ color-singletness restriction at the partition function level. This causes a metastable state of a heavy glueball within the temperature range $T_s < T < T_c$. A heavy glueball is what Abbas[4] has obtained in his study of the Dark Matter problem in the early Universe scenario. We have therefore confirmed his prediction here. Quite clearly we expect our calculations to have implications for the early universe QCD phase transition and the QGP phase transition. These problems are currently under investigation by us.

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FIGURE CAPTIONS

Figure 1: The graphical solutions of Eq.(15) at various temperature is shown. The curve labelled by **(a)** is the temperature independent plot of $y = x^4$ vs x . The other curves are plots of $y = ax^3 + g$ vs x labelled by **(b), (c), (d), (e)** corresponding to temperature $T = 210$ MeV ($< T_s$), $T = T_s$ (217.218 MeV for $B^{1/4} = 250$ MeV), $T = 240$ MeV, and $T_c = 253.3$ MeV respectively. Intersections of curve **(a)** with others are denoted by solid circles represent physical solutions. The parameter value $C = 6.0$ is used.

Figure 2: The solutions of eq.(15) (i.e. the bag radius) as a function of temperature is shown. Here $C = 6.0$, $B^{\frac{1}{4}} = 250$ MeV, $T_s = 217.218$ MeV , and $T_c = 253.3$ MeV .

TABLE CAPTIONS

Table 1

Results for the case $T_s < T < T_c$ with $B^{\frac{1}{4}} = 250$ Mev, $T_s = 217.218$ MeV and $T_c = 253.3$ MeV

Table 1

T MeV	Lower	State	Higher	State
	Radius(fm)	Mass(GeV)	Radius(fm)	Mass(GeV)
218	0.45	0.79	2.87	203
220	0.45	0.79	1.81	50.7
230	0.45	0.79	0.98	8.0
240	0.46	0.83	0.74	3.5
250	0.48	0.94	0.59	1.76